A NEW APPROACH TO NONRENORMALIZABLE THEORIES

Ruggero Ferrari
Daniele Bettinelli and Andrea Quadri
Dip. di Fisica, Università degli Studi di Milano and INFN, Sez. di Milano

Abstract

The problem of constructing a sensible physical theory out of a nonrenormalizable classical action is discussed and instanced in the paradigmatic case of the nonlinear sigma model.

At variance with the approach of algebraic renormalization, the perturbative expansion in $\hbar$ is not constructed on the most general classical action, where each new independent divergent amplitude induces a novel parameter. Instead a solution is searched where the number of free parameters is fixed and all the requirements associated to symmetry properties, defining equations, locality, physical unitarity are met. Once these conditions are obeyed, the theory can be tested experimentally.

This program can be realized in the case of the nonlinear sigma model. This is achieved if one looks at the fields as parameters of a gauge field of zero strength (flat connection).

In this important example, some technical items are developed which will be useful in other more complex models (e.g. massive Yang-Mills theory) as local functional equations, hierarchy, weak power-counting.

The final result is a model depending on two parameters: the v.e.v. of the spontaneous breakdown of symmetry and the scale of the radiative corrections.

The subtraction of the divergences is performed in dimensional regularization by minimal subtraction on the properly normalized 1-PI amplitudes.

1 Introduction

The Standard Model, as a quantum field theory, relies on two solid pillars: the absence of anomalies and the power-counting renormalizability (PCR). The first guarantees the validity of Slavnov- Taylor identity and consequently the physical unitarity of the model. PCR seems more a manifestation of the technical difficulty to deal with models containing interaction terms of dimension higher than four, but it implies the necessity of extra physical modes as the Higgs boson.

In a series of papers we investigated the possibility of constructing a consistent theory out of a non polynomial classical action [1) - 5)]. By "consistency" we mean a perturbation theory in $\hbar$ where divergences are removed by local counterterms order by order in a symmetric fashion, i.e. by preserving the defining equations and the invariance properties under the relevant local and global transformations on the fields. By "theory" we mean a calculation procedure which starts from a classical effective action depending on a finite number of parameters and where the radiative corrections are controlled by an extra mass scale and the physical unitarity is guaranteed. We stress that in our approach: i) no extra free parameters are introduced in connection with new divergent amplitudes and ii) predictivity is implied by the presence of a finite number of free parameters.

By our approach we abandon the view of renormalization as a replacement of the bare parameters of the Lagrangian in favor of the physical ones. We look at the removal of the divergences as a pure mathematical problem: if a procedure exists that respects all the requirements listed above, than we have a consistent theory. The question of uniqueness of the procedure might be posed, but we shall not discuss this item here.

We apply this approach to the nonlinear sigma model. A quantum field theory based on the Feynman rules of the nonlinear sigma model is plagued not only by the presence of an infinite number of superficially divergent amplitudes but also by the fact that the divergences are not chiral invariant. These difficulties are present already at the one loop level, as has been widely discussed in the existing literature [6)- 13)].

We can go through successfully with our program in the case of the nonlinear sigma model, because the pion fields can be seen as parameters of a gauge field of zero strength (flat connection). Within this formulation one can use the powerful methods of the gauge theories. In particular a local functional equation for the generating functional of the 1 PI amplitudes [1] can be introduced in order to define the model itself. The equation stems from the invariance under local chiral (left) transformations of the Haar measure in the path integral. This formulation overcomes the difficulty due to the lack of chiral symmetry of the divergences. The subtraction of the divergences is performed in dimensional regularization by using minimal subtraction on the properly normalized 1-PI amplitudes. In the present work we discuss this subtraction procedure and compare it with algebraic renormalization [14]- [17] and with effective field theory approach [18].

The construction of the perturbative series starts from the Feynman rules of the nonlinear sigma model. The radiative corrections are regularized by continuation in the dimensions. The strategy by which the divergences are removed in the limit $D = 4$ makes use of two important properties of the functional equation, that are duly discussed in Ref. [1], [2] and [3]: i) hierarchy and ii) Weak Power Counting (WPC). As summarized in Section 2 the functional equation has a rigid hierarchy structure in the loop expansion: all amplitudes involving the pion fields (descendant amplitudes) are derived from those involving only insertions of the flat connection ($F_{a\mu}$) and the order parameter (the constraint ϕ_0), the ancestor amplitudes. The important consequence of this fact comes from the second property: the WPC. At each order of the loop expansion the number of divergent ancestor amplitudes is finite, since the superficial divergence of a graph is (N_J and N_K are the number of flat connection and order parameter insertions)

$$(D-2)n + 2 - N_J - 2N_{K_0}. \tag{1}$$

Thus at each loop order all amplitudes are made finite by a finite number of subtractions. Moreover WPC remains valid only if one does not introduce terms of higher dimensions in the tree-level Feynman rules. These facts suggest our subtraction strategy: if one finds a way to perform subtractions without introducing free parameters for higher dimensions counterterms in the tree level Feynman rules, then one gets a consistent theory with a finite number of physical parameters. The subtraction strategy is suggested by the functional equation itself. The violation of the equation at $n-$th order, when the counterterms are introduced up to order $n-1$, has simple pole structure in $D-4$. Then minimal subtraction automatically restores the functional equation.

These subtraction rules are at variance not only with the fundamentals of algebraic renormalization, but also with the effective theory approach and with the strategy proposed by the renormalization in the 'modern' sense of [19]. In fact in the present approach the subtraction procedure is a fundamental part of the construction of the theory and in practical way it means that one cannot introduce new vertex Feynman rules in connection with counterterms. Thus the subtraction procedure we are proposing has some aspects that make it look as an *ad hoc* strategy in the choice both of the regularization method and of the counterterms. In this work we try to show that if one wants a (predictive) theory by starting from the tree level nonlinear sigma model, then our proposal is a consistent and sensible strategy.

The question whether this subtraction can be performed by means of other regularization schemes has been considered. Limited results have been achieved. By using the renormalized linear sigma model in the limit of large coupling constant one can get, after subtraction of divergent terms, the nonlinear sigma model we are proposing (one loop has been checked in ref. [20]). This requires a fine tuning in the finite subtractions and consequently there is no evidence for a particular advantageous choice in the finite subtraction as in dimensional regularization. In order to study this issue it is very useful to consider the most general solution allowed by the linearized homogeneous functional equation. At one loop this means seven arbitrary coefficients associated to the invariants reported in Sect. 5. The same pattern is present in other regularization procedures as Pauli-Villars.
The issue of the number of physical parameters in a theory which is not renormalizable by power-counting has been discussed several times in the recent literature. In [21] it has been proposed to introduce a framework for reducing [22] the infinite number of free parameters to a smaller, eventually finite, one. A similar strategy has been advocated in [23] in the context of Wilson's approach to renormalization [24].

In this paper we argue that the lore, by which an infinite number of experiments is required in order to fix the counterterms for a nonrenormalizable theory, stems from an inappropriate use of the point of view of the algebraic renormalization to theories that cannot be treated according to such a procedure.

In the case of the nonlinear sigma model the theory is defined through the effective action Γ which has to obey a nonlinear <u>local</u> functional equation. At the one loop level the counterterms $\widehat{\Gamma}^{(1)}$ obey a linearized form of the same equation. These counterterms have a particular feature: they are not present in the vertex functional $\Gamma^{(0)}$ at the tree level. Some of them do not obey the *nonlinear* defining functional equation. Others modify in a substantial way the unperturbed space of states (by introducing ghost states associated to kinetic terms in $\Box^2$). Finally there are some that could be introduced in the vertex functional $\Gamma^{(0)}$ at the tree level, since they obey the defining local functional equation, but they would spoil the WPC. Thus the procedure of assigning free parameters to the counterterms is not viable.

We discuss also the possibility of assigning free parameters to the counterterms at the one loop level. We argue that this strategy is not sustainable from the physical point of view, since parameters should enter in the zero loop vertex functional $\Gamma^{(0)}$. We stress this fundamental point: the parameters of the classical action might differ from the physical parameters of the zero loop vertex functional. The presence of a vacuum state that induces a reshuffling of the perturbative expansion (spontaneous symmetry breaking) is one example where such a distinction is essential.

After we have excluded free parameters in association to the counterterms, the question remains of the number of independent parameters. One parameter is present in $\Gamma^{(0)}$; for instance, the vacuum expectation value of ϕ_0. However an extra mass parameter can be introduced in order to perform dimensional subtraction. We argue that this parameter has the very important role of fixing the scale of the radiative corrections. One can formulate the model in such a way that the dimensional subtraction scale appears as a front factor of the whole set of Feynman rules. The final consequence of this requirement is that our subtraction procedure yields a nonlinear sigma model depending on only two parameters, e.g. the v.e.v. of the spontaneous breakdown and the dimensional subtraction scale.

The present paper is devoted to a detailed illustration of the above mentioned facts and it is written in the spirit of a novel view on those nonrenormalizable theories that can be consistently subtracted (i.e. symmetrically and locally). The discussion is done at the one loop level, but the necessary tools for the extension at higher order are also provided. In particular we discuss the equation obeyed by the counterterms at any order in the loop expansion.

2 The Nonlinear Sigma Model

The D-dimensional classical action of the nonlinear sigma model in the flat connection formalism [1)] is

$$\Gamma^{(0)} = \Lambda^{D-4} \int d^D x \, \frac{v^2}{8} \Big(F_a^\mu - J_a^\mu \Big)^2 + \int d^D x \, K_0 \phi_0 . \tag{2}$$

The flat connection is

$$\begin{aligned} F^\mu &= F_a^\mu \frac{1}{2} \tau_a = i \Omega \partial_\mu \Omega^\dagger \\ \Omega &= \frac{1}{v} (\phi_0 + i \tau_a \phi_a) \end{aligned} \tag{3}$$

where v is the v.e.v. of the order parameter field ϕ_0, $\vec{J}_\mu$ is the background connection and K_0 is the source of the constraint ϕ_0 of the nonlinear sigma model

$$\phi_0 = \sqrt{v^2 - \vec{\phi}^2} \, . \tag{4}$$

2.1 Divergences in the nonlinear Sigma Model

When one considers the perturbative solution in $\hbar$ of (2)

$$\begin{aligned} &\Lambda^{D-4} \int d^D x \, \frac{v^2}{8} \Big(F_a^\mu - J_a^\mu \Big)^2 + \int d^D x \, K_0 \phi_0 \\ &= \Lambda^{D-4} \int d^D x \left(\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + \frac{1}{2} \frac{(\phi_b \partial^\mu \phi_b)(\phi_c \partial_\mu \phi_c)}{v^2 - \phi^2} \right), \end{aligned} \tag{5}$$

it is soon evident that the non polynomial interaction term gives rise to infinitely many divergent amplitudes already at one loop as in Fig. 1. Explicit calculations then show that the global chiral symmetry is broken by the divergent parts.

The usual approach to remove the divergences, by means of free parameters, fails to provide a theory for many reasons: i) one needs an infinite number of free parameters; ii) the new terms associated to the divergent amplitudes worsen the ultraviolet behavior of the tree level action; iii) the global chiral symmetry is broken. In one word: it is not a theory.

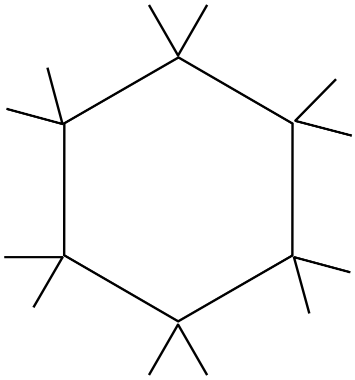

Figure 1: *Example of divergent 1-loop graph.*

2.2 Flat Connection Approach

The formulation of the nonlinear sigma model formulated in terms of a flat connection $F_\mu = i\Omega\partial_\mu\Omega^\dagger$ allows us to use the methods of local gauge theories. The path integral measure $\mathcal{D}[\phi]\delta(\phi_0^2 + \vec{\phi}^2 - v^2)$ is invariant under <u>local</u> chiral transformations induced by left multiplication on Ω by SU(2) matrices

$$U(\omega) \simeq 1 + \frac{i}{2}\tau_a\omega_a. \tag{6}$$

Then $\Gamma^{(0)}$ obeys a D-dimensional local functional equation associated to the local chiral transformations. This equation is required to be valid for the effective action on the basis of a path integral formulation of the model

$$-\partial^\mu\frac{\delta\Gamma}{\delta J_a^\mu} - \epsilon_{abc}J_b^\mu\frac{\delta\Gamma}{\delta J_c^\mu} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta\Gamma}{\delta\phi_b} + \frac{1}{2}\phi_a K_0 + \frac{1}{2}\frac{\delta\Gamma}{\delta K_0}\frac{\delta\Gamma}{\delta\phi_a} = 0\,. \tag{7}$$

The equation is local (no x-integration). The generating functional of the Green functions obeys the corresponding equation

$$\Big(\partial^\mu\frac{\delta}{\delta J_a^\mu} + \epsilon_{abc}J_b^\mu\frac{\delta}{\delta J_c^\mu} + \frac{1}{2}\epsilon_{abc}K_b\frac{\delta}{\delta K_c} - \frac{1}{2}K_0\frac{\delta}{\delta K_a} + \frac{1}{2}K_a\frac{\delta}{\delta K_0}\Big)Z = 0. \tag{8}$$

The spontaneous breakdown of the global chiral symmetry is fixed by the boundary condition

$$\left.\frac{\delta\Gamma}{\delta K_0}\right|_{\vec\phi=\vec J_\mu=K_0=0} = v. \tag{9}$$

It will be required that these equations ((7), (8)and (9)) remain valid also for the subtracted amplitudes (symmetric subtraction).

The tree level amplitudes are fixed by the conditions

$$\begin{aligned}
&\frac{\delta^2\Gamma^{(0)}}{\delta J_a^\mu(x)\delta J_b^\nu(y)} = \frac{v^2\Lambda^{D-4}}{4} g_{\mu\nu}\delta_{ab}\delta_D(x-y)\\
&\frac{\delta^2\Gamma^{(0)}}{\delta K_0(x)\delta K_0(y)} = 0\\
&\frac{\delta^2\Gamma^{(0)}}{\delta K_0(x)\delta J_b^\nu(y)} = 0.
\end{aligned} \tag{10}$$

The naïve Feynman rules in D dimensions given implicitly in eq. (5) yield amplitudes that solve eqs. (7) and (8) at any order of the perturbative expansion. This property has been conjectured in ref. [1] and proved in ref. [4].

3 Hierarchy

The non linearity of the equation (7) is responsible for many peculiar facts. In particular by eq.(9) $\frac{\delta\Gamma}{\delta K_0}$ is invertible as a formal power series. Therefore by using eq.(7) all amplitudes involving the $\vec\phi$ fields (descendants) can be derived from those of $\vec F_\mu$ and ϕ_0 (ancestors), i.e. the functional derivatives with respect to $\vec J_\mu$ and K_0 (hierarchy).

Hierarchy is very important for the procedure of divergences subtraction. First of all one needs to make finite only the ancestor amplitudes. The subtraction of the divergences for the descendant amplitudes will follows automatically, since the lasts are functions of the previouses. Second, one can try to exploit the properties of the eq. (7), e.g. symmetries, in order to devise the counterterms as functionals of the ancestor variables $\vec J_\mu, K_0$, where the $\vec\phi$ can be introduced in agreement with the results of hierarchy. This amounts to integrate functionally eq. (7) as done in Ref. [5].

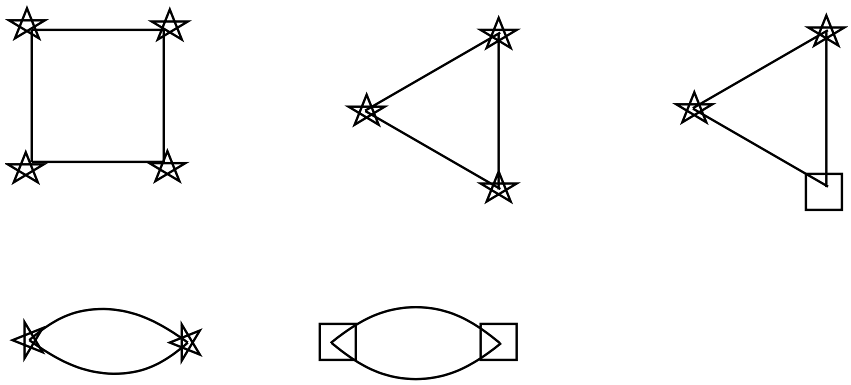

Figure 2: *Ancestor divergent 1-loop graphs. Stars are $\vec{J}_\mu$ and boxes are K_0.*

4 Weak Power-Counting

Hierarchy reduces drastically the number of independent divergent amplitudes. According to the Feynman rules provided by eq. (5) the superficial degree of divergence of a n_L-th loop amplitude involving N_J insertions of the flat connection and N_{K_0} insertions of the nonlinear sigma model constraint is

$$\delta = (D-2)n_L + 2 - N_J - 2N_{K_0} \, . \tag{11}$$

For instance at one loop the ancestor divergent graphs are those depicted in Fig. 2.

The proof of eq. (11) is straightforward in D dimensions and without counterterms. The crucial question is whether this relation is still valid after the introduction of the counterterms necessary in order to take the limit $D = 4$. Details will not be given here. However we can argue that the relation in eq. (11) is not modified by our subtraction procedure, since the divergences are not removed by renormalization of novel free parameters in the tree level effective action. Instead the counterterms are subtraction rules at the appropriate $\hbar$ order. It can be verified that this fact does not alter eq.(11) since the worsened ultraviolet behavior of the counterterms is exactly compensated by the corresponding reduced number of loops in order to get the given power of $\hbar$. By this argument the finger-counting of the superficial degree of divergence as in eq.

(11) becomes a theorem: the power-counting is stable under the subtraction procedure.

5 Subtraction of Divergences at $D = 4$

Subtractions at $D = 4$ are performed in dimensional regularization. The procedure can be better described if the external source K_0 has a canonical dimension independent of D. The effective action (2) at the tree level takes the form

$$\Gamma^{(0)} = \Lambda^{D-4} \int d^D x \left(\frac{v^2}{8} \left(F_a^\mu - J_a^\mu \right)^2 + K_0 \phi_0 \right). \tag{12}$$

Eqs. (7), (8) and (9) are accordingly modified. For instance

$$-\partial^\mu \frac{\delta \Gamma}{\delta J_a^\mu} - \epsilon_{abc} J_b^\mu \frac{\delta \Gamma}{\delta J_c^\mu} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta \Gamma}{\delta \phi_b} + \frac{1}{2} \Lambda^{D-4} \phi_a K_0 + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma}{\delta K_0} \frac{\delta \Gamma}{\delta \phi_a} = 0 \tag{13}$$

and

$$\left. \frac{\delta \Gamma}{\delta K_0} \right|_{\vec{\phi} = \vec{J}_\mu = K_0 = 0} = v \Lambda^{D-4}. \tag{14}$$

The counterterms $\mathcal{M}^{(j)}$ will be introduced as local functionals of $\vec{J}_\mu$, K_0 and $\vec{\phi}$ which now have canonical dimensions independent of D. The whole set of Feynman rules $\widehat{\Gamma}$ is then written in the form

$$\widehat{\Gamma} = \Lambda^{D-4} \int d^D x \left(\frac{v^2}{8} \left(F_a^\mu - J_a^\mu \right)^2 + K_0 \phi_0 + \sum_{j>0} \mathcal{M}^{(j)} \right) \tag{15}$$

where j denotes the order $\hbar^j$. The dependence on D is confined in the prefactor Λ^{D-4}. Our *Ansatz* on the subtraction procedure is on the form of $\mathcal{M}^{(j)}$: they contain only pole parts in $D - 4$. No finite parts are introduced, since these cannot be fixed *a priori*. As a consequence of the *Ansatz* the pole parts $\mathcal{M}^{(j)}$ at order j are given by the Laurent expansion of

$$-\frac{1}{\Lambda^{D-4}} \Gamma^{(j)} \tag{16}$$

where $\Gamma^{(j)}$ is collection of the 1-PI amplitudes at order j after all the counterterms have been introduced up to order $j - 1$.

Whether this subtraction procedure is consistent with the defining equations (13) and (14) is a highly non trivial question.

5.1 Subtraction at one Loop

At one loop level the counterterms (obtainable from the pole parts of the vertex functional $\Gamma^{(1)}$) obey a linearized form of the local equation eq.(13)

$$\mathcal{S}_a(\widehat{\Gamma}^{(1)}) = \Big[\frac{1}{2\Lambda^{D-4}}\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0} + \frac{1}{2}\phi_0\frac{\delta}{\delta\phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b}$$
$$-\partial^\mu\frac{\delta}{\delta J^\mu_a} - \epsilon_{abc}J^\mu_b\frac{\delta}{\delta J^\mu_c}\Big]\widehat{\Gamma}^{(1)} = 0\,. \qquad (17)$$

It is easy to trace in eq. (17) the transformations induced through the dependence on $\vec{J}_\mu$ and $\vec{\phi}$.

5.2 Subtraction at n-Loops

The proof for any order in $\hbar$ has been given in Reference [4]. We outline some of the steps of the proof since they are illuminating for our subtracting strategy.

One can give a diagrammatic proof of the following relation between the local functional equation (8) for the generating functional Z and the equation for the counterterms (quantum action principle).

$$\Big(-\partial^\mu\frac{\delta}{\delta J^\mu_a} - \epsilon_{abc}J^\mu_b\frac{\delta}{\delta J^\mu_c} + \frac{\Lambda^{D-4}}{2}K_0\frac{\delta}{\delta K_a} - \frac{1}{2\Lambda^{D-4}}K_a\frac{\delta}{\delta K_0} - \frac{1}{2}\epsilon_{abc}K_b\frac{\delta}{\delta K_c}\Big)Z$$
$$= i\left(-\partial^\mu\frac{\delta\widehat{\Gamma}}{\delta J^\mu_a} - \epsilon_{abc}J^\mu_b\frac{\delta\widehat{\Gamma}}{\delta J^\mu_c} - \frac{\Lambda^{D-4}}{2}K_0\phi_a + \frac{1}{2\Lambda^{D-4}}\frac{\delta\widehat{\Gamma}}{\delta K_0}\frac{\delta\widehat{\Gamma}}{\delta\phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta\widehat{\Gamma}}{\delta\phi_b}\right)\cdot Z\,, \qquad (18)$$

where the dot indicates the insertion of the local operators, i.e. operators to be put inside the path-integral. Since the L.H.S. is zero by eq. (8) we can generalize the condition (17) for the counterterms to

$$\mathcal{S}_a(\widehat{\Gamma}^{(n)}) = \Big[\frac{1}{2\Lambda^{D-4}}\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0} + \frac{1}{2}\phi_0\frac{\delta}{\delta\phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b}$$
$$-\partial^\mu\frac{\delta}{\delta J^\mu_a} - \epsilon_{abc}J^\mu_b\frac{\delta}{\delta J^\mu_c}\Big]\widehat{\Gamma}^{(n)} = -\frac{1}{2\Lambda^{D-4}}\sum_{j=1}^{n-1}\frac{\delta\widehat{\Gamma}^{(j)}}{\delta K_0}\frac{\delta\widehat{\Gamma}^{(n-j)}}{\delta\phi_a}\,. \qquad (19)$$

The above equation is the key point of the subtraction procedure. Assume that counterterms have been introduced up to order $n-1$. At order n eq. (13) is broken by the fact that the n-th order counterterm $\widehat{\Gamma}^{(n)}$ is absent. Thus we have

$$\Big(-\partial^\mu \frac{\delta}{\delta J_a^\mu} - \epsilon_{abc} J_b^\mu \frac{\delta}{\delta J_c^\mu} + \frac{1}{2}\epsilon_{abc}\phi_c \frac{\delta}{\delta \phi_b} + \frac{1}{2}\phi_0 \frac{\delta}{\delta \phi_a}$$
$$+\frac{1}{2\Lambda^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0}\Big)\Gamma^{(n)} + \frac{1}{2\Lambda^{(D-4)}} \sum_{j=1}^{n-1} \frac{\delta}{\delta K_0}\Gamma^{(n-j)} \frac{\delta}{\delta \phi_a}\Gamma^{(j)}$$
$$= \frac{1}{2\Lambda^{(D-4)}} \sum_{j=1}^{n-1} \frac{\delta}{\delta K_0}\widehat{\Gamma}^{(n-j)} \frac{\delta}{\delta \phi_a}\widehat{\Gamma}^{(j)}. \tag{20}$$

The above equation is the origin of our subtraction procedure. If one divides both members by $\Lambda^{(D-4)}$ the R.H.S. contains only pole parts according to the *Ansatz* in eq. (15) (no finite parts are left over). Thus removing only the pole parts in both members divided by $\Lambda^{(D-4)}$ reestablishes the local functional equation (13).

It should be stressed that the prescription of removing the poles from $\Lambda^{(4-D)}\Gamma^{(n)}$ is a well defined choice on the finite parts. This choice depends on the value of Λ, which becomes a free parameter of the theory. For $n > 0$ the subtracted effective action is

$$\Gamma^{(n)} + \widehat{\Gamma}^{(n)} \equiv \Big(\Gamma^{(n)} + \Lambda^{(D-4)} \int d^D x \mathcal{M}^{(n)}(x)\Big)\Big|_{D=4}$$
$$= \Lambda^{(D-4)}\Big(\frac{1}{\Lambda^{(D-4)}}\Gamma^{(n)} + \int d^D x \mathcal{M}^{(n)}(x)\Big)\Big|_{D=4}$$
$$= \Big(\frac{1}{\Lambda^{(D-4)}}\Gamma^{(n)} + \int d^D x \mathcal{M}^{(n)}(x)\Big)\Big|_{D=4} \tag{21}$$

6 One Loop invariants

It is very interesting how the hierarchy is implemented in the counterterms. Since the counterterms obey the eq. (19), in the one loop case one has to find all local independent solutions of the linearized eq. (17). The invariants must have dimension 4 in order to meet the condition $\delta \geq 0$ in eq. (11).

For sake of simplicity, instead of $\mathcal{S}_a$ of eq.(17) we use the operator [2)]

$$s \equiv \int d^D x\, \omega_a \mathcal{S}_a = \int d^D x \,\Big[\frac{1}{2}\Big(\omega_a \phi_0 + \epsilon_{ajk}\phi_j \omega_k\Big)\frac{\delta}{\delta \phi_a}$$

$$+\frac{1}{2\Lambda^{D-4}}\omega_a\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0}+\Big(\partial^\mu\omega_a+\epsilon_{aij}J_i^\mu\omega_j\Big)\frac{\delta}{\delta J_a^\mu}\Big] \tag{22}$$

and consider the Legendre transform

$$K_a\equiv-\frac{\delta}{\delta\phi_a}\Gamma^{(0)}. \tag{23}$$

Then we use following identities

$$\begin{aligned}
&s\ \Gamma^{(0)}=-\int d^Dy\,\omega_a\frac{1}{2}\Big(\phi_0K_a-\Lambda^{D-4}\phi_aK_0\Big)\\
&[\frac{\delta}{\delta\phi_a(x)},s]=\omega_k(x)\frac{1}{2}\Big(-\delta_{kb}\frac{\phi_a(x)}{\phi_0(x)}+\epsilon_{bak}\Big)\frac{\delta}{\delta\phi_b(x)}\\
&-\frac{1}{2\Lambda^{D-4}}\int d^Dy\omega_b(y)\frac{\delta K_b(y)}{\delta\phi_a(x)}\frac{\delta}{\delta K_0(y)}
\end{aligned} \tag{24}$$

and one gets

$$\begin{aligned}
&s\ K_a(x)=-s\ \frac{\delta}{\delta\phi_a}\Gamma^{(0)}=[\frac{\delta}{\delta\phi_a(x)},s]\Gamma^{(0)}-\frac{\delta}{\delta\phi_a(x)}\ s\ \Gamma^{(0)}\\
&=-\omega_k(x)\frac{1}{2}\Big(-\delta_{kb}\frac{\phi_a(x)}{\phi_0(x)}+\epsilon_{bak}\Big)K_b(x)\\
&-\frac{1}{2}\omega_b(x)\frac{\delta K_b(y)}{\delta\phi_a(x)}\phi_0(x)\\
&+\omega_b(x)\frac{1}{2}\Big(-\frac{\phi_a(x)}{\phi_0(x)}K_b+\phi_0\frac{\delta K_b(y)}{\delta\phi_a(x)}+\Lambda^{D-4}\delta_{ab}K_0\Big)\\
&=\frac{1}{2}\epsilon_{abk}\omega_kK_b+\frac{1}{2}\Lambda^{D-4}\omega_aK_0
\end{aligned} \tag{25}$$

and

$$s\ K_0=-\frac{1}{2\Lambda^{D-4}}\omega_aK_a. \tag{26}$$

Then

$$\begin{aligned}
&\overline{K}_0\equiv K_0\phi_0+\frac{1}{\Lambda^{D-4}}K_a\phi_a\\
&=K_0\phi_0-v^2\phi_a\frac{\partial}{\partial\phi_a}(\vec F_\mu-\vec J_\mu)^2+\frac{K_0}{\phi_0}\phi_a\phi_a\\
&=\frac{v^2K_0}{\phi_0}-v^2\phi_a\frac{\partial}{\partial\phi_a}(\vec F_\mu-\vec J_\mu)^2
\end{aligned} \tag{27}$$

is invariant under s:

$$s\ \overline{K}_0=0. \tag{28}$$

Other invariants can be obtained by using *bleached* variables [5]. One constructs the bleached variables by means of the transformation properties of $\vec{J}_\mu$ and Ω under the operator s of eq. (22). Accordingly the following variables

$$\left[\Omega^\dagger \left(J_\mu - i\Omega\partial_\mu\Omega^\dagger\right)\Omega\right]_{ab} \tag{29}$$

are invariant under s-transformations for any a and b, where a, b are right-indices. The same is true for any point derivative of the above operators. Notice that

$$\begin{aligned}
&\partial_\nu\left[\Omega^\dagger \left(J_\mu - i\Omega\partial_\mu\Omega^\dagger\right)\Omega\right]_{ab} \\
&= \left\{\Omega^\dagger\left[-iF_\nu\left(J_\mu - F_\mu\right) + \partial_\nu\left(J_\mu - F_\mu\right) + i\left(J_\mu - F_\mu\right)F_\nu\right]\Omega\right\}_{ab} \\
&= \left\{\Omega^\dagger D_\nu[F]\left(J_\mu - F_\mu\right)\Omega\right\}_{ab}
\end{aligned} \tag{30}$$

The invariant solutions can be obtained from monomial constructed in terms of the invariant variables in eqs. (27), (29) and (30). The $SU(2)_R$ indices have to be saturated. The invariant solutions of the linearized functional equation which enter at the one loop level are of dimension four. We list here the basis obtained in Reference [2] by using the component notation.

$$\begin{aligned}
\mathcal{I}_1 &= \int d^Dx \left[D_\mu(F-J)_\nu\right]_a \left[D^\mu(F-J)^\nu\right]_a , \\
\mathcal{I}_2 &= \int d^Dx \left[D_\mu(F-J)^\mu\right]_a \left[D_\nu(F-J)^\nu\right]_a , \\
\mathcal{I}_3 &= \int d^Dx \, \epsilon_{abc}\left[D_\mu(F-J)_\nu\right]_a \left(F_b^\mu - J_b^\mu\right)\left(F_c^\nu - J_c^\nu\right) , \\
\mathcal{I}_4 &= \int d^Dx \left(\overline{K}_0\right)^2 , \\
\mathcal{I}_5 &= \int d^Dx \left(\overline{K}_0\right)\left(F_b^\mu - J_b^\mu\right)^2 , \\
\mathcal{I}_6 &= \int d^Dx \left(F_a^\mu - J_a^\mu\right)^2\left(F_b^\nu - J_b^\nu\right)^2 , \\
\mathcal{I}_7 &= \int d^Dx \left(F_a^\mu - J_a^\mu\right)\left(F_a^\nu - J_a^\nu\right)\left(F_{b\mu} - J_{b\mu}\right)\left(F_{b\nu} - J_{b\nu}\right) ,
\end{aligned} \tag{31}$$

where D_μ denotes the covariant derivative w.r.t $F_{a\mu}$:

$$D_{ab\mu} = \delta_{ab}\partial_\mu + \epsilon_{acb}F_{c\mu}\,. \tag{32}$$

In terms of ϕ fields the flat connection is

$$F_a^\mu = \frac{2}{v^2}\big(\phi_0\partial^\mu\phi_a - \partial^\mu\phi_0\phi_a + \epsilon_{abc}\partial^\mu\phi_b\phi_c\big)\,. \tag{33}$$

By dimensional arguments one expects that at one loop the counterterms (the $1/(D-4)$ pole parts) are linear combinations of $\mathcal{I}_1 \ldots \mathcal{I}_7$. In Ref. [2] the linear combination is explicitly evaluated, by using the ancestor graphs in Fig. 2. One gets

$$\begin{aligned}\hat{\Gamma}^{(1)} \quad = \quad & \frac{\Lambda^{D-4}}{D-4}\Big[-\frac{1}{(4\pi)^2}\frac{1}{12}\Big(\mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3\Big) + \frac{1}{(4\pi)^2}\frac{1}{48}\Big(\mathcal{I}_6 + 2\mathcal{I}_7\Big) \\ & + \frac{1}{(4\pi)^2}\frac{3}{2}\mathcal{I}_4 + \frac{1}{(4\pi)^2}\frac{1}{2}\mathcal{I}_5\Big]\,. \end{aligned} \tag{34}$$

On dimensional grounds other solutions of eq.(17) are excluded, e.g.

$$\int d^Dx\overline{K}_0. \tag{35}$$

The invariants $\mathcal{I}_1 \ldots \mathcal{I}_7$ implement the hierarchy for the counterterms. Their ϕ dependence through F_μ and ϕ_0 fixes completely all the counterterms for the descendant amplitudes at one loop level. There exists an identity which turns out to be interesting when one derives the descendant counterterms from $\hat{\Gamma}^{(1)}$. One verifies that

$$2(\mathcal{I}_1 - \mathcal{I}_2) - 4\mathcal{I}_3 + \mathcal{I}_6 - \mathcal{I}_7 = \int d^Dx\, G_{a\mu\nu}[J]G_a^{\mu\nu}[J]\,, \tag{36}$$

Since the field strength $G_a^{\mu\nu}[J]$ depends only on the field strength of the external source $J_{a\mu}$, this particular combination has no descendants.

7 Parameters Fixing

In this section we show that we cannot introduce at the tree level new Feynman vertices associated to the one-loop counterterms if we want to produce a sensible and consistent theory.

Minimal subtraction is of course a very interesting option in order to make finite the perturbative series. The proof, that this subtraction algorithm is symmetric (i.e. eq. (7) is stable), makes the procedure consistent. Thus this theory can be tested by experiments.

A frequent objection to the present proposal of making finite a nonrenormalizable theory is that one needs seven parameter-fixing appropriate measures in order to evaluate the coefficients of $\mathcal{I}_1 \ldots \mathcal{I}_7$. This objection is legitimate if the above mentioned invariants are action-like. As one should do in power counting renormalizable theories, according to algebraic renormalization. Here the situation is more involved. This is evident if we paraphrase the problem in the following way. Can we introduce at the tree level the seven invariants with arbitrary coefficients and treat them as *bona fide* interaction terms intervening in the loop expansion as the original one provided in $\Gamma^{(0)}$ of eq. (2)? The answer to this question is in general negative. If one allows this modification of the unperturbed effective action, the one loop corrections will be modified by extra terms generated by the newly introduced Feynman rules, thus bringing to a never ending story.

In particular the introduction at tree level of the vertices described by the invariants in eq.(31) implies new Feynman rules which invalidate the weak power-counting [2)]. The superficial degree of divergence of the ancestor amplitudes is not any more given by eq. (11). As a direct consequence of the violation of the weak power-counting, already at one loop the number of divergent ancestor amplitudes is infinite.

A closer look to $\mathcal{I}_1 \ldots \mathcal{I}_7$ shows that there are also other reasons that forbid the use of some of these invariants as unperturbed effective action terms. $\mathcal{I}_1, \mathcal{I}_2$ can be introduced into $\Gamma^{(0)}$ without breaking eq. (7). However they modify the spectrum of the unperturbed states (by introducing negative norm states) through kinetic terms with four derivatives, unless they appear in the combination $\mathcal{I}_1 - \mathcal{I}_2$. $\mathcal{I}_4, \mathcal{I}_5$ cannot be introduced into $\Gamma^{(0)}$ because they violate eq. (7).

8 Finite Subtractions

After we excluded the possibility of introducing in the tree level effective action the invariants $\mathcal{I}_1 \ldots \mathcal{I}_7$, there is still the possibility to use them for a finite, in principle arbitrary, renormalization. I.e. in the book keeping of the Feynman rules one could enter new terms

$$\hbar \sum_j \lambda_j \int d^D x\, \mathcal{I}_j(x)\,, \tag{37}$$

where we have explicitly exhibited the $\hbar$ factor in order to remind that these vertices are of first order in $\hbar$ expansion. λ_j are arbitrary real parameters.

More explicitly we can tell the story in the following way. The subtraction of the poles in $D-4$ requires a series of counterterms of the form (37) where the coefficients carry the pole factor $1/(D-4)$. Then it seems reasonable to use these extra degrees of freedom as free parameters.

In the PCR case the fixing of the finite parts of the symmetric counterterms can be seen as a way to introduce the renormalization by a reset of the parameters entering into the classical action. The situation is clearly different in the present case, since the invariants $\mathcal{I}_1, \ldots, \mathcal{I}_7$ are not action-like and therefore the additional parameters λ_j can be introduced only as quantum corrections.

The meaning of this latter procedure, outside an effective field theory approach [18], seems to us rather unclear from the physical point of view, since independent parameters are used in the radiative corrections.

The alternative approach (which we favor) is to assume that symmetrically subtracted theories (and not only PCR theories) should obey the principle ruling PCR models, namely that parameters have to be introduced *ab initio* in the vertex functional $\Gamma^{(0)}$ at the tree level.

9 Minimal Subtraction vs. Effective Field Theory Approach

The above discussion illustrates the fact that we face an antinomy. From a mathematical point of view, finite subtractions as in eq. (37) are allowed and yield the most general solution to the subtraction procedure. From a physical point of view, free parameters as λ_j cannot be introduced in the radiative corrections.

In order to shed some light on this issue it is helpful to compare in some detail minimal subtraction with the effective field theory approach.

9.1 Minimal Subtraction

In minimal subtraction we use pure pole subtraction in order to make the theory finite in $D = 4$. Even with this clear cut strategy, still there is some freedom left connected to the presence of a second scale parameter in the Feynman rules in dimensional renormalization. The tree level effective action in eq. (12) has been written in order to evidence the dependence from the radiative scale parameter Λ.

The (non trivial) finite parts of the subtractions are governed by the sole front factor $\Lambda^{-(D-4)}$ in eq. (12). The resulting amplitudes depend on the parameters v and Λ. The last one is not present in the classical action at $D = 4$: it sneaks in as a scale of the radiative corrections.

A similar mechanism has a renowned antecedent in the theory of Lamb shift [25)], where the radiative corrections due to the excited state transitions need a ultraviolet cut-off which is not present at the lowest level of the theory of the Hydrogen atom.

A comment is in order here. In PCR theories the free parameters in the classical action can be fixed by a set of normalization conditions at a given mass scale Λ. Moreover, a shift in Λ is reabsorbed by a shift in the same free parameters entering into the classical action (renormalization group).

On the contrary in the NLSM a shift in Λ cannot be compensated by a shift in v. Therefore Λ has to be treated as a second independent free parameter (in addition to v) to be determined through the fit with the experimental data.

9.2 Effective field theory approach

The subtraction scheme based on minimal subtraction is symmetric and fulfills the WPC. Moreover from the above discussion it turns out that it admits only two physical parameters, both of them entering in the tree level D-dimensional vertex functional fixing the tree-level Feynman rules.

In addition this scheme fulfills weak independence on the regularization, namely the Green functions of minimal subtraction can be reproduced in any symmetric regularization by a fine-tuning of the coefficients of the relevant invariants $\mathcal{I}_j$. This follows since the WPC is regularization-independent.

As such, minimal subtraction looks like a viable proposal for making the theory finite in $D = 4$.

Let us compare it with the effective field theory approach. In this latter case the coefficients of the invariants $\mathcal{I}_j$ are considered as independent free parameters to be fixed by a suitable set of normalization conditions. Since the number of invariants $\mathcal{I}_j$ allowed by the WPC increases with the number of loops, there are infinitely many normalization conditions to be given (effective field theory).

In the effective field theory approach strong independence on the regularization holds: in fact the results of any symmetric regularization, in the presence of whatever choice of normalization conditions, can be reproduced in a different symmetric regularization scheme by a fine-tuning of the coefficients of the relevant invariants $\mathcal{I}_j$. The equivalence of arbitrary regularizations requires to make full use of the infinite number of free parameters (with the prescribed grading in $\hbar$) mathematically allowed by the subtraction procedure and the functional equation.

It is clear that the effective field theory approach is incompatible with minimal subtraction. In fact in the latter only two free parameters are at disposal, and hence the infinite set of normalization conditions which have to be fixed according to the effective approach cannot be reproduced.

10 Conclusions

In this work we have outlined a new approach to nonrenormalizable field theories. One abandons the point of view of algebraic renormalization where each independent divergent amplitude necessitates of a free parameter in the effective action at the tree level. Instead one looks for a subtraction procedure, where the number of free parameters is finite and which is respectful of the relevant properties of model, as defining equations, locality of the counterterms, symmetry properties and physical unitarity (symmetric subtraction). We have examined the $SU(2)$ nonlinear sigma model, where the fields are parameters of a gauge field with zero strength (flat connection). The invariance of the path integral measure under local left $SU(2)$ transformations gives a local functional equation that contains all the properties for a symmetric subtraction of the divergences. One is hierarchy among the 1-PI amplitudes where the ancestors determine completely the descendants, i.e. those containing at least one chiral field. The ancestor variables are the flat connection and the constrained field. The second important property is the weak power-counting theorem for the ancestor amplitudes: at each order in $\hbar$ the number of independent superficially divergent amplitudes is finite. This properties suggests the strategy of removal of the divergences, which consists in the minimal subtraction of the poles in $D-4$ of the properly normalized amplitudes. The procedure is consistent: the local functional equation, locality of counterterms, symmetry properties are not modified by the counterterms. The strategy has been recently applied to the massive Yang-Mills theory [26].

The final output of this method of removing the divergences is a computation strategy where only a finite number of free parameters appear. In the case of the nonlinear sigma model they are the v.e.v. of the order parameter field in the tree level effective action and the scale of the radiative corrections which enters through the dimensional regularization.

We have briefly argued that algebraic renormalization cannot be used both because the number of free parameters is infinite and because the counterterms of order n modify the superficial degree of divergence of the terms of order less or equal n.

Moreover we have avoided the possibility to introduce free parameters at any given order in $\hbar$, as it is done in effective field theory. This is possible from a mathematical point of view, since any local independent solution of the homogeneous equation can carry its own free parameter. However we have the prejudice that all parameters should be present in the tree level effective action. Radiative corrections are expected to modify the tree level amplitudes but not to introduce new degrees of freedom, with the exeption of the scale which naturally enters in the subtraction procedure.

11 Acknowledgments

One of us (R.F.) is very much indebted to A.A. Slavnov for stimulating discussions.

References

1. R. Ferrari, JHEP **0508** (2005) 048 [arXiv:hep-th/0504023].

2. R. Ferrari and A. Quadri, Int. J. Theor. Phys. **45** (2006) 2497 [arXiv:hep-th/0506220].

3. R. Ferrari and A. Quadri, JHEP **0601** (2006) 003 [arXiv:hep-th/0511032].

4. D. Bettinelli, R. Ferrari and A. Quadri, "A comment on the renormalization of the nonlinear sigma model," arXiv:hep-th/0701197.

5. D. Bettinelli, R. Ferrari and A. Quadri, JHEP **0703** (2007) 065 [arXiv:hep-th/0701212].

6. J. M. Charap, Phys. Rev. D **2** (1970) 1554.

7. J. M. Charap, Phys. Rev. D **3**, 1998 (1971).

8. J. Honerkamp and K. Meetz, Phys. Rev. D **3** (1971) 1996.

9. I. S. Gerstein, R. Jackiw, S. Weinberg and B. W. Lee, Phys. Rev. D **3** (1971) 2486.

10. G. Ecker and J. Honerkamp, Nucl. Phys. B **35** (1971) 481.

11. L. D. Faddeev and A. A. Slavnov, Lett. Nuovo Cimento **8** (1973) 117.

12. L. Tataru, Phys. Rev. D **12** (1975) 3351.

13. T. Appelquist and C. W. Bernard, Phys. Rev. D **23** (1981) 425.

14. K. Symanzik, Commun. Math. Phys. **16** (1970) 48.

15. O. Piguet and S. P. Sorella, Lect. Notes Phys. **M28**, 1 (1995).

16. R. Ferrari and P.A. Grassi, Phys. Rev. **D60**: 65010 (1999).

17. R. Ferrari, P.A. Grassi and A. Quadri, Phys. Lett. **B472**, 346-356 (2000).

18. J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142; J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

19. J. Gomis and S. Weinberg, Nucl. Phys. B **469** (1996) 473 [arXiv:hep-th/9510087].

20. D. Bettinelli, R. Ferrari and A. Quadri, "The hierarchy principle and the large mass limit of the linear sigma model," [arXiv:hep-th/0611063], to appear in Int. J. Theor. Phys.

21. D. Anselmi, Class. Quant. Grav. **12** (1995) 319 [arXiv:hep-th/9407023]; Class. Quant. Grav. **20** (2003) 2355 [arXiv:hep-th/0212013]; JHEP **0507** (2005) 077 [arXiv:hep-th/0502237]; JHEP **0508** (2005) 029 [arXiv:hep-th/0503131]; D. Anselmi and M. Halat, JHEP **0601** (2006) 077 [arXiv:hep-th/0509196].

22. W. Zimmermann, Commun. Math. Phys. **97** (1985) 211.

23. J. Kubo and M. Nunami, Eur. Phys. J. C **26** (2003) 461 [arXiv:hep-th/0112032].

24. K. G. Wilson, Phys. Rev. B **4** (1971) 3174; Rev. Mod. Phys. **47** (1975) 773; K. G. Wilson and J. B. Kogut, Phys. Rept. **12** (1974) 75.

    P. Breitenlohner and D. Maison, Commun. Math. Phys. **52** (1977) 39; Commun. Math. Phys. **52** (1977) 55; Commun. Math. Phys. **52** (1977) 11.

25. H. A. Bethe, Phys. Rev. **72**, 339 (1947).

26. D. Bettinelli, R. Ferrari and A. Quadri, arXiv:0705.2339 [hep-th].